The explicit formula of flat Lagrangian H-umbilical submanifolds in quaternion Euclidean spaces

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Abstract

In [4], there exist nonflat Lagrangian H-umbilical submanifolds in $\mathbf{H^n}$: Lagrangian pseudo-sphere and a quaternion extensor of the unit hypersphere of $\mathbf{E^n}$. In this paper, using the idea of twisted product, we investigate the flat Lagrangian H-umbilical submanifolds in quaternion Euclidean space $\mathbf{H^n}$.

1 Introduction

We begin with the following results from [1]. B.Y. Chen introduced the notion of Lagrangian H-umbilical submanifold in $\mathbf{C^n}$ and classified the Lagrangian H-umbilical sumaifolds M in $\mathbf{C^n}$:flat, Lagrangian pseudo-sphere, or complex extensor of the unit hypersphere of $\mathbf{E^n}$. Moreover, he also obtained the following results for the flat case. Let M be a simply-connected open portion of the twisted product manifold ${}_f\mathbf{R} \times \mathbf{E^{n-1}}$ with the metric

$$g = f^2 dx_1^2 + \sum_{j=2}^{n} dx_j^2,$$

where $f = \beta(x_1) + \sum_{j=2}^{n} \alpha_j(x_1) x_j$ for some real valued functions β and $\alpha_2, ..., \alpha_n$. There exists a *unique* Lagrangian isometric immersion $L_f : M \to C^n$ with the second fundamental form

$$h(e_1, e_1) = \lambda J e_1, h(e_1, e_j) = h(e_j, e_k) = 0, 2 \leq j, k \leq n,$$

where $\lambda = \frac{1}{f}, e_1 = \lambda \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial x_2}, \cdots, e_n = \frac{\partial}{\partial x_n}$. Furthermore, if $n \geq 3$, and $L : M \to C^n$ is a Lagrangian H-umbilical isometric immersion of a flat manifold into C^n without totally geodesic points, then M is an open portion of a twisted product manifold ${}_f\mathbf{R} \times \mathbf{E^{n-1}}$ described as above. Later, in [2], B.Y. Chen obtained the explicit description of this isometric immersion using the idea of a special Legendre curve.
Based on the facts above, we can impose the following questions.
Question 1. Are there any flat Lagrangian H-umbilical submanifolds in quaternion Euclidean spaces?
I could answer this question in theorem 4.1 [4] and get more details in section 3. In fact, besides this flat submanifolds, there exists a Lagrangian pseudosphere in $\mathbf{C^n}$ and a quaternionic extensor of the unit hypersphere of $\mathbf{E^n}$.
Question 2. If exists, what is the explicit description of this isometric immersion?

We remark here that we follow the notations and definitions given in [4].

2 Preliminaries

We have the following existence and uniqueness theorems.

Theorem 2.1 *Let M^n be a simply connected Riemannian n-manifold and $\sigma_i (i = 1, 2, 3)$ be TM-valued symmetric bilinear forms on M such that*
(a) $< \sigma_i(X, Y), Z >$ is totally symmetric for $i = 1, 2, 3$
(b) $(\nabla_X \sigma_i)(Y, Z) - \sigma_j(X, \sigma_k(Y, Z)) + \sigma_k(X, \sigma_j(Y, Z))$ is totally symmetric, where $(\nabla_X \sigma_i)(Y, Z) = \nabla_X \sigma_i(Y, Z) - \sigma_i(\nabla_X Y, Z) - \sigma_i(Y, \nabla_X Z)$ and $(i, j, k) = (1, 2, 3), (2, 3, 1),$ or $(3, 1, 2)$.
(c) $R(X, Y)Z = \sum_{i=1}^{3} \{\sigma_i(\sigma_i(Y, Z), X) - \sigma_i(\sigma_i(X, Z), Y)\}$
Then there exists a Lagrangian isometric immersion $x : M^n \to \mathbf{H^n}$ whose second fundamental form $h(X, Y) = I\sigma_1(X, Y) + J\sigma_2(X, Y) + K\sigma_3(X, Y)$.

Theorem 2.2 *Let $L_1, L_2 : M^n \to \mathbf{H^n}$ be two Lagrangian isometric immersion of a Riemannian n-manifold with the second fundamental forms h^1 and h^2, respectively. If, for $i = 1, 2, 3$*

$$< h^1(X, Y), \pi_i L_{1\star} Z > = < h^2(X, Y), \pi_i L_{2\star} Z >$$

for all tangent vector fields X, Y, Z on M^n, and $\pi_i = I, J$ or K, then there exists an isometry ϕ of $\mathbf{H^n}$ such that $L_1 = L_2 \circ \phi$.

Now, here is the sketch of the proof of theorem 2.1.
<u>Proof of theorem 2.1</u> We define a bundle NM over M by $NM = TM \oplus TM \oplus TM$, $\pi_i : TM \to NM$, where $\pi_1(X) = (X, 0, 0), \pi_2 = (0, X, 0), \pi_3 = (0, 0, X)$. We also define a connection on NM by

$$\begin{aligned} \nabla^{\perp}_X(\pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3) &= \pi_1 \nabla_X Y_1 + \pi_3 \sigma_2(X, Y_1) - \pi_2 \sigma_3(X, Y_1) \\ &+ \pi_2 \nabla_X Y_2 - \pi_3 \sigma_1(X, Y_2) + \pi_1 \sigma_3(X, Y_2) \\ &+ \pi_3 \nabla_X Y_3 + \pi_2 \sigma_1(X, Y_3) - \pi_1 \sigma_2(X, Y_3). \end{aligned}$$

Then we can define the second fundamental form $h : TM \times TM \to NM$ by $h(X, Y) = \pi_1 \sigma_1(X, Y) + \pi_2 \sigma_2(X, Y) + \pi_3 \sigma_3(X, Y)$. Its corresponding Weingarten maps are given by $A_{\pi_i X} Y = \sigma_i(X, Y), i = 1, 2, 3$. Then the straightforward long calculations show that this setting satisfies the Gauss, Codazzi and Ricci equations. Applying the Existence theorem, there exists an isometric immersion $x : M^n \to \mathbf{E^{4n}}$ with the normal bundle NM, second fundamental form h, normal connection $\nabla^{\perp}$ and Weingarten operator A. Let's define three endomorphisms I, J and K on $\mathbf{E^{4n}} = \mathbf{TM} + \mathbf{NM}$ as below:

$$\begin{array}{llll} I|_{TM} = \pi_1 TM & I|_{\pi_1 TM} = -TM & I|_{\pi_2 TM} = \pi_3 TM & I|_{\pi_3 TM} = -\pi_2 TM \\ J|_{TM} = \pi_2 TM & J|_{\pi_2 TM} = -TM & J|_{\pi_3 TM} = \pi_1 TM & J|_{\pi_1 TM} = -\pi_3 TM \\ K|_{TM} = \pi_3 TM & K|_{\pi_3 TM} = -TM & K|_{\pi_1 TM} = \pi_2 TM & K|_{\pi_2 TM} = -\pi_1 TM \end{array}$$

It is easy to check that these three endomorphisms are almost complex structures satisfying:

$$I^2 = J^2 = K^2 = -1, IJ = K, JI = -K, JK = I, KJ = -I, KI = J, IK = -J$$

Using these defintions, we can easily verify that the second fundamental form h is now given by

$$h(X, Y) = I\sigma_1(X, Y) + J\sigma_2(X, Y) + K\sigma_3(X, Y)$$

Finally, we must show that I, J and K are parallel. For X, Y tangent vector fields to M, we get

$$\begin{aligned} (\tilde{\nabla}_X I)Y &= -A_{IY} X + \nabla^{\perp}_X(IY) - I\nabla_X Y - Ih(X, Y) \\ &= -\sigma_1(X, Y) + \pi_1 \nabla_X Y - \pi_2 \sigma_3(X, Y) + \pi_3 \sigma_2(X, Y) \\ &\quad -\pi_1 \nabla_X Y - Ih(X, Y) = 0. \end{aligned}$$

Similarly, we can also show that $(\tilde{\nabla}_X J)Y = 0, (\tilde{\nabla}_X K)Y = 0$ and $(\tilde{\nabla}_X \varphi)(\pi_i Y) = 0$, where $\varphi = I, J$ or K and $i = 1, 2, 3$. Therefore, we can identify $\mathbf{E^{4n}}, I, J$ and K with $\mathbf{H^n}$ and we can easily see that this isometric immersion x is Lagrangian.

Now, we recall a definition of twisted product [1]. Let N_1, N_2 be two Riemannian manifolds with Riemannian metrics g_1, g_2, respectively and f a positive function on $N_1 \times N_2$. Then the metric $g = f^2 g_1 + g_2$ is called a twisted product metric on $N_1 \times N_2$. The manifold $N_1 \times N_2$ with the twisted product metric $g = f^2 g_1 + g_2$ is called a twisted product manifold, which is denoted by ${}_f N_1 \times N_2$. The function f is called the twisting function of the twisted product manifold.

3 Main results

In order to characterize the flat Lagrangian submaifold into quaternion Euclidean spaces, we need the quaternion version of special Legendre curve in $S^{n-1} \in \mathbf{C^n}$ introduced by B.Y. Chen in his paper [2].

Let $z : I \rightarrow S^{4n-1} \subset \mathbf{H^n}$ be a unit speed curve in the unit hypersphere centered at the origin in $\mathbf{H^n}$ satisfying the following condition: $< z'(s), iz(s) >= < z'(s), jz(s) >=< z'(s), kz(s) >= 0$ identically. Hence $z(s), iz(s), jz(s), kz(s), z'(s), iz'(s), jz'(s), kz'(s)$ are orthonormal vector fields defined along the curve. Thus, there exists normal vector fields $P_3, P_4, ..., P_n$ such that $z(s), iz(s), jz(s), kz(s), z'(s), iz'(s), jz'(s), kz'(s), P_3, iP_3, jP_3, kP_3, ..., P_n, iP_n, jP_n, kP_n$ form an orthonormal frame field along the Legendre curve. Using these orthonormal vector fields, $z''(s)$ can be written as

$$\begin{aligned}(1)\ z''(s) &= i\alpha(s)z'(s) + j\beta(s)z'(s) + k\gamma(s)z'(s) - z(s) - \sum_{l=3}^{n} a_l(s)P_l(s) \\ &+ \sum_{l=3}^{n} b_l(s)iP_l(s) + \sum_{l=3}^{n} c_l(s)jP_l(s) + \sum_{l=3}^{n} d_l(s)kP_l(s),\end{aligned}$$

where $\alpha, \beta, \gamma, a_l, b_l, c_l$ and d_l are all real valued functions. The Legendre curve $z = z(s)$ is called a special Legendre curve in $S^{4n-1} \subset \mathbf{H^n}$ if the expression

(1) is simplified to

$$z''(s) = i\alpha(s)z'(s) + j\beta(s)z'(s) + k\gamma(s)z'(s) - z(s) - \sum_{l=3}^{n} a_l(s)P_l(s)$$

for some parallel normal vector fields $P_3, P_4, ..., P_n$ along the curve.
We note here that any Legendre curve in $S^7 \subset \mathbf{H^2}$ is special.

Theorem 3.1 *(a) Let M^n be a simply connected open portion of the twisted product manifold ${}_f\mathbf{R} \times \mathbf{E^{n-1}}$ with twisted product metric*

$$g = f^2 dx_1^2 + \sum_{j=2}^{n} dx_j^2$$

where $f^2 = f_1^2 + f_2^2 + f_3^2$ for three arbitrary functions on M such that $\frac{f}{f_1}, \frac{f}{f_2}, \frac{f}{f_3}$ are functions of only x_1 and also

$$f(x_1, ..., x_n) = \beta(x_1) + \sum_{j=2}^{n} \alpha_j(x_1)x_j$$

for some functions $\beta, \alpha_1, ..., \alpha_n$ of x_1.
Then, up to rigid motions of $\mathbf{H^n}$, there is a unique Lagrangian isometric immersion $L_f : M \to \mathbf{H^n}$ without totally geodesic points whose second fundamental form satisfies

$$h(e_1, e_1) = \lambda_1 Ie_1 + \lambda_2 Je_1 + \lambda_3 Ke_1, h(e_1, e_j) = h(e_j, e_k) = 0 \quad j, k \geq 2,$$

where $e_1 = \dfrac{1}{f}\dfrac{\partial}{\partial x_1}, e_i = \dfrac{\partial}{\partial x_i}, i \geq 2, \lambda_j = \dfrac{1}{f_j}, j = 1, 2, 3$

(b) Suppose $L : M^n \to \mathbf{H^n}(\mathbf{n \geq 3})$ is a Lagrangian H-umbilical isometric immersion of a flat manifold into $\mathbf{H^n}$ without totally geodesic points. Then M is an open portion of a twisted product ${}_f\mathbf{R} \times \mathbf{E^{n-1}}$ with twisted product metric $g = f^2dx_1^2 + dx_2^2 + \cdots + +dx_n^2$ and twisted product fuction given in statement (a). Up to rigid motions of $\mathbf{H^n}$, L is the uniquely given by the L_f above in (a). However,

(i) If $\alpha_2 = \cdots = \alpha_n = 0$, i.e. f is a function of x_1 only, then L is given by $L(x_1, ..., x_n) = D(x_1) + \sum_{j=2}^{n} c_j x_j$ which is a Lagrangian cylinder over a curve $D(x_1)$ whose rulings are $(n-1)$ planes parallel to $x_2 \cdots x_n$-planes

in $\mathbf{H^n}$.

(ii) Otherwise, by doing some change of variables $t = \int_0^{x_1} \alpha_2(x)dx, u_2 = x_2, \cdots, u_n = x_n$, *$L$ is given by*

$$L(t, u_2, \cdots, u_n) = u_2 z(t) + \sum_{j=3}^{n} u_j P_j(t) + \int^t b(t) z'(t) dt$$

for some $\mathbf{H^n}$ *valued functions* $P_3, ..., P_n$ *of* t, *where the twisted product metric* $g = \tilde{f}^2 dt^2 + du_2^2 + \cdots + du_n^2$, *and twisted product function* $\tilde{f}(t, u_2, ..., u_n) = b(t) + u_2 + \sum_{j=3}^{n} a_j(t) u_j$. *Here,* $z = z(t)$ *is a special Legendre curve in* S^{4n-1}.

Proof. (a) Define three symmetric bilinear forms σ_1, σ_2 and σ_3 on M^n by $\sigma_1(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) = \frac{f}{f_1}\frac{\partial}{\partial x_1}, \sigma_2(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) = \frac{f}{f_2}\frac{\partial}{\partial x_1}, \sigma_3(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) = \frac{f}{f_3}\frac{\partial}{\partial x_1}$ and all other are zero. Then $< \sigma_i(X, Y), Z >$ is totally symmetric in X, Y, and Z. Using the twisted product metric given, we have

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} = \frac{(f)_1}{f}\frac{\partial}{\partial x_1} - f \textstyle\sum_2^n (f)_k \frac{\partial}{\partial x_k}, \\
&\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_i} = \frac{(f)_i}{f}\frac{\partial}{\partial x_1}, \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = 0, i, j, k = 2, ..., n,
\end{aligned}$$

where $(f)_j = \frac{\partial f}{\partial x_j}$ for $j = 1, ..., n$. We note here that $(f)_1 = \beta'(x_1) + \sum_{j=2}^{n} \alpha_j'(x_1) x_j$, and $(f)_j = \alpha_j(x_1)$ for $j = 2, ..., n$. The long straightforward computations show that all three conditions of theorem 2.1 are satisfied. Therefore, there exists a Lagrangian isometric immersion $L_f : M^n \to \mathbf{H^n}$ whose second fundamental form is given by $h(X, Y) = I\sigma_1(X, Y) + J\sigma_2(X, Y) + K\sigma_3(X, Y)$. Up to rigid motions of $\mathbf{H^n}$, it is unique by theorem 2.2. However, if we put $e_1 = \frac{1}{f}\frac{\partial}{\partial x_1}, e_i = \frac{\partial}{\partial x_i}, i \geq 2$, then $h(e_1, e_1) = \lambda_1 I e_1 + \lambda_2 J e_1 + \lambda_3 k e_1, \lambda_i = \frac{1}{f_i}, i = 1, 2, 3$. and $h(e_i, e_j) = 0$ for $i, j \geq 2$.

(b) The second fundamental form for L is given by

$$\begin{aligned}
h(e_1, e_1) &= \lambda_1 I e_1 + \lambda_2 J e_1 + \lambda_3 K e_1 \\
h(e_1, e_j) &= \mu_1 I e_j + \mu_2 J e_j + \mu_3 K e_j, j \geq 2 \\
h(e_i, e_i) &= \mu_1 I e_1 + \mu_2 J e_1 + \mu_3 K e_1, i \geq 2 \\
h(e_j, e_k) &= 0, \quad j \neq k \geq 2,
\end{aligned} \tag{2}$$

for some real valued functions λ_i,and $\mu_i, i = 1, 2, 3$ with respect to some orthonormal frame fields $\{e_1, e_2, ..., e_n\}$. Since $n \geq 3$, we can compute the

sectional curvature of the plane spanned by e_i, and e_j for $i \neq j \geq 2$ which implies that $\mu_1 = \mu_2 = \mu_3 = 0$. Now, the second fundamental form given above becomes

$$
\begin{aligned}
h(e_1, e_1) &= \lambda_1 I e_1 + \lambda_2 J e_1 + \lambda_3 K e_1 \\
h(e_i, e_j) &= o \text{ for all } i, j \text{ except } i = 1, \text{ and } j = 1
\end{aligned} \tag{3}
$$

By Codazzi equation, we get

$$
\begin{aligned}
e_i(\lambda_1) &= \omega_1^i(e_1)\lambda_1 \\
e_i(\lambda_2) &= \omega_1^i(e_1)\lambda_2 \\
e_i(\lambda_3) &= \omega_1^i(e_1)\lambda_3
\end{aligned} \tag{4}
$$

Also, we have

$$\nabla_{e_i} e_1 = 0. \tag{5}$$

Let D and $D^{\perp}$ be the distributions spanned by e_1 and $\{e_2, ..., e_n\}$, respectively. Since D is 1 dimensional, D is integrable. Also, $D^{\perp}$ is integrable because of (5). Moreover, the leaves of D and $D^{\perp}$ are totally geodesic submanifolds of $\mathbf{H^n}$. Since D and $D^{\perp}$ are integrable and perpendicular, there exist local coordinates $\{x_1, ..., x_n\}$ such that $\frac{\partial}{\partial x_1}$ spans D and $\{\frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\}$ spans $D^{\perp}$. Since D is 1 dimensional, we can choose x_1 such that $\frac{\partial}{\partial x_1} = |\lambda| e_1, |\lambda|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Therefore, by Hiepko's theorem([3]), M is isometric to an open portion of the twisted product manifold ${}_f I \times \mathbf{E^{n-1}}$ with the twisted product metric $g = f^2 dx_1^2 + \sum_{j=2}^n dx_j^2, f = |\lambda|$.

We will consider the following case according to (4).

(**case** 1) $\omega_i^i(e_1) = 0$ for all $i \geq 2$

This condition implies that $e_j(\lambda_1) = e_j(\lambda_2) = e_j(\lambda_3) = 0$ for all $j \geq 2$. It means that the twistor function f is a function depending only on x_1. Using the twisted product given above, we have

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} &= \frac{f'}{f} \frac{\partial}{\partial x_1} \\
\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_i} &= \nabla_{\frac{\partial}{\partial x_{1j}}} \frac{\partial}{\partial x_k} = 0, \text{ for } i, j, k = 2, ..., n
\end{aligned} \tag{6}
$$

Combining (3), (6) and Gauss' formula yield

$$L_{x_1 x_1} = (\frac{f'}{f} + (i\lambda_1 + j\lambda_2 + k\lambda_3)|\lambda|) L_{x_1} \tag{7}$$

$$
\begin{aligned}
L_{x_1x_j} &= 0\\
L_{x_jx_k} &= 0
\end{aligned}
$$

By the third equation of (7), we get $L(x_1,...,x_n) = D(x_1) + \sum_{j=2}^{n} P_j(x_1)x_j$ for some $\mathbf{H^n}$-valued functions $D, P_2, ..., P_n$. Using the second equation of (7), we can find that $P_j(x_1) = c_j$ where c_j's are constant vectors in $\mathbf{H^n}$. Therefore, L is a Lagrangian cylinder over a curve $D = D(x_1)$ whose rulings are $(n-1)$ plane parallel to $x_2 \cdots x_n$ plane in $\mathbf{H^n}$.
(**case** 2) $\omega_i^i(e_1) \neq 0$ for some $i \geq 2$
If we assume $\lambda_1, \lambda_2, \lambda_3$ are positive, then we have $e_j(\ln\lambda_1) = e_j(\ln\lambda_2) = e_j(\ln\lambda_3)$ for $j \geq 2$ and thus $\frac{f}{\lambda_1}, \frac{f}{\lambda_2}, \frac{f}{\lambda_3}$ are all functions only depending on x_1. Using the twisted product metric g, we get

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} &= \frac{f_1}{f}\frac{\partial}{\partial x_1} - f\sum_{k=2}^{n} f_k \frac{\partial}{\partial x_k}\\
\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_i} &= \frac{f_i}{f}\frac{\partial}{\partial x_1},\\
\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} &= 0, \text{ for } i,j,k = 2,...,n
\end{aligned}
\tag{8}
$$

By (8), we can compute the Riemannain curvature tensor $R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j})$ and using the fact that M^n is flat, we have $f_{jk} = 0$ for all $j, k \geq 2$ which implies that f is given by

$$
f(x_1,...,x_n) = \beta(x_1) + \sum_{j=2}^{n} \alpha_j(x_1)x_j
$$

for some functions $\beta, \alpha_1, ..., \alpha_n$ of x_1.
Therefore, by (a), M^n is a Lagrangian submanifold described in the statement (a).
More explicitly, we can describe the manifold M^n as stated in the theorem by doing the same computation done in [2].

Next, we also have the following theorem for surfaces.

Theorem 3.2 *Let $L : M^2 \to \mathbf{H^2}$ be a Lagrangian H-umbilical isometric immersion of a flat surface into $\mathbf{H^2}$ without totally geodesic points, then we*

have the following cases.
(a) M^2 is a Lagrangian cylinder in $\mathbf{H^2}$ i. e. $L(x,y) = D(x) + cy$ for a curve $D = D(x)$ and a constant vector c in $\mathbf{H^2}$.
(b) L is given by $L(x,y) = D(x) + P(x)y$ for some $\mathbf{H^2}$-valued curves D and P. In fact, here $P = P(x)$ is a special Legendre curve in $S^7 \subset \mathbf{H^2}$.
(c) $L(x,y) = f(x)A(y)$, where f is a $\mathbf{H}$-valued function and A is a curve in $\mathbf{H^2}$. In fact, it is a cone over a curve A in $\mathbf{H^2}$ plane.

PROOF. We can have (a) and (b) if $\mu_1 = \mu_2 = \mu_3 = 0$ in (2). Its proof is same as theorem 3.1. For the case if $|\mu| \neq 0$, we can also prove in the same way as theorem 3.1 and get the result (c).

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